# Generalized geometry of pseudo Riemannian manifolds and generalized debar operator

# Generalized geometry of pseudo-Riemannian manifolds and generalized $\overline{\partial}-$operator

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ABSTRACT. Let $(M, g, \nabla)$ be a pseudo-Riemannian manifold with a torsion free linear connection and let $J^g$ be the generalized complex structure on $M$ defined by $g$, [13], [14]. We prove that in the case $J^g$ is $\nabla-$integrable the $\pm i-$eigenbundles of $J^g$, $E^{1,0}_{J^g}$, $E^{0,1}_{J^g}$, are complex Lie algebroids. Moreover $E^{0,1}_{J^g}$ and $\left(E^{1,0}_{J^g}\right)^*$ are canonically isomorphic thus we define the concept of generalized $\overline{\partial} - operator$ of $(M, g, \nabla)$ and we describe a class of generalized holomorphic sections of $T(M) \oplus T^*(M)$. Also we relate Lie bialgebroid property of $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ to conditions on the metric $g$ in the case of affine Hessian manifolds. [1]

## 1 Introduction

Let $(M, g)$ be a smooth pseudo-Riemannian manifold, let $T(M)$ be the tangent bundle, let $T^*(M)$ be the cotangent bundle and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of $M$. Generalized complex structures were introduced by Nigel Hitchin in [6], and further investigated by Marco Gualtieri in [8], in order to unify symplectic and complex geometry. In this paper we consider a more general concept of generalized complex structure introduced in [13], [14] and also studied in [15], [1]. In the previous papers [13], [14], we defined a generalized complex structure of $M$ as a complex structure on $E$ and we studied some classes of such structures, in particular calibrated complex structures with respect to the canonical symplectic structure, $(\ ,\ )$, of $E$. Using a torsion free linear connection, $\nabla$, on $M$ we introduced a bracket, $[\ ,\ ]_\nabla$, on sections of $E$, the corresponding concept of $\nabla-$integrability for complex structures and we studied integrability conditions. Moreover in [14] we described a large class of almost complex structures on cotangent bundles of manifolds endowed with a torsion free linear connection, induced by generalized complex structures and we proved that, in the case $\nabla$ has zero curvature, a $\nabla-$integrable generalized complex structure on $M$ defines a complex structure on $T^*(M)$. In this paper we concentrate on the canonical generalized complex structure defined by

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$g$, $J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}$. We prove that in the case $J^g$ is $\nabla-$integrable the $\pm i-$eigenbundles of $J^g$, $E^{1,0}_{J^g}$, $E^{0,1}_{J^g}$, are complex Lie algebroids. Then we observe that the natural symplectic structure of $T(M) \oplus T^*(M)$ defines a canonical isomorphism between $E^{0,1}_{J^g}$ and $\left(E^{1,0}_{J^g}\right)^*$ and this allows us to define the *generalized* $\overline{\partial}_{J^g}-$*operator* on $M$. We prove that in the case $J^g$ is $\nabla-$integrable we get $\left(\overline{\partial}_{J^g}\right)^2 = 0$, moreover $\overline{\partial}_{J^g}$ is the exterior derivative of the Lie algebroid $E^{1,0}_{J^g}$, in particular $\left(C^\infty\left(\wedge^\bullet\left(E^{1,0}_{J^g}\right)\right), \wedge, \overline{\partial}_{J^g}, [\ ,\ ]_\nabla\right)$, where $\wedge$ is the Schouten bracket, is a differential Gerstenhaber algebra, [9], [19]. We also study generalized holomorphic sections and we prove that, for any $X \in T(M)\otimes\mathbb{C}$, the section $\sigma = X + ig(X) \in C^\infty\left(\left(E^{1,0}_{J^g}\right)^*\right)$ is generalized holomorphic if and only if $g(X)$ is a Lagrangian submanifold of $T^*(M)$, in particular in the case $(M, g, \nabla)$ is an affine Hessian manifold we describe a large class of local generalized holomorphic sections of $\left(E^{1,0}_{J^g}\right)^*$. Finally we study the Lie bialgebroid condition on the two algebroids in duality $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ and we prove that, unlike the case of generalized complex structures in the sense of Hitchin, this gives restrictions on $g$. The paper is organized as in the following. In section 2 we introduce preliminary material: we describe the main geometrical properties of the generalized tangent bundle and of generalized complex structures; moreover we recall the basic definitions in the setting of complex Lie algebroids and Lie bialgebroids. Original results are concentrated in section 3: the definition of the generalized $\overline{\partial}_{J^g}-$ *operator* and its properties; in particular we relate generalized holomorphic sections of $E^{0,1}_{J^g}$ to Lagrangian submanifolds of $T^*(M)$. Section 4 is devoted to Hessian manifold because they occur as interesting examples in our context.

# 2 Preliminary Material

## 2.1 Geometry of the generalized tangent bundle

Let $M$ be a smooth manifold of real dimension $n$ and let $E = T(M) \oplus T^*(M)$ be the *generalized tangent bundle* of $M$, we recall the main geometric properties of $E$.

Smooth sections of $E$ are elements $X + \xi \in C^\infty(E)$ where $X \in C^\infty(T(M))$ is a vector field and $\xi \in C^\infty(T^*(M))$ is a $1-$ form.

$E$ is equipped with a natural *symplectic structure* defined by:

$$(X + \xi, Y + \eta) = -\frac{1}{2}(\xi(Y) - \eta(X)) \tag{1}$$

and a natural *indefinite metric* defined by:

$$< X + \xi, Y + \eta > \ = -\frac{1}{2}(\xi(Y) + \eta(X)) \tag{2}$$

$< \, , \, >$ is non degenerate and of signature $(n, n)$.

A linear connection, $\nabla$, on $M$, defines, in a canonical way, a bracket on $C^{\infty}(E)$, $[\ ,\ ]_{\nabla}$, as follows:

$$[X+\xi, Y+\eta]_{\nabla} = [X, Y] + \nabla_X \eta - \nabla_Y \xi. \tag{3}$$

The following holds:

**Lemma 2.** ([13]) *For all $X, Y \in C^{\infty}(T(M))$, for all $\xi, \eta \in C^{\infty}(T^*(M))$ and for all $f \in C^{\infty}(M)$ we have*:

1. $[X+\xi, Y+\eta]_{\nabla} = -[Y+\eta, X+\xi]_{\nabla}$ ,
2. $[f(X+\xi), Y+\eta]_{\nabla} = f\,[X+\xi, Y+\eta]_{\nabla} - Y(f)(X+\xi)$ ,
3. *Jacobi's identity holds for $[\ ,\ ]_{\nabla}$ if and only if $\nabla$ has zero curvature.*

Moreover:

**Proposition 2.** ([14]) *Let $\nabla$ be a connection on $M$ then there is a bundle morphism*

$$\Phi^{\nabla} : T(M) \oplus T^*(M) \to T(T^*(M)) \tag{4}$$

*which is an isomorphism on the fibres and such that*:

1. $\Phi^{\nabla}$ *identifies $T^*(M)$ with vertical vectors that is*:

$$\left(\Phi^{\nabla}\right)^{-1}(\ker \pi_*) = T^*(M));$$

2. $\pi_* \circ \Phi^{\nabla}_{|T(M)} = I_{|T(M)}$,

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3. $\left(\Phi^{\nabla}\right)^*(\Omega) = -2(\ ,\ )$ *if and only if $\nabla$ has zero torsion*;
4. $\left(\Phi^{\nabla}\right)([\ ,\ ]_{\nabla}) = \left[\Phi^{\nabla}\ , \Phi^{\nabla}\ \right]$ *if and only if $\nabla$ has zero curvature.*

where, we denoted by $\pi_*$ the tangent map of

$$\pi : T^*(M) \to M, \tag{5}$$

$$\pi_* : T(T^*(M)) \to T(M) \tag{6}$$

$$(\pi_*(A))(f) = A(f \circ \pi)$$

for all $A \in T(T^*(M))$ and for all $f \in C^{\infty}(M)$,

and $\Omega = d\theta$ where $\theta$ is the *Liouville's 1-form* defined by:

$$\theta(A) = p(A)(\pi_*(A)), \text{ for all } A \in T(T^*(M)). \tag{7}$$

In this paper we consider the following concept of generalized complex structure, introduced in [13], [14] and also studied in [15], [1] :

**Definition 3.** A *generalized complex structure on $M$* is an endomorphism $J : E \to E$ such that $J^2 = -I$.

A pseudo-Riemannian metric on $M$, $g$, defines, in a natural way, a complex structure $J^g$ on $E$ by:

$$J^g(X+\xi) = -g^{-1}(\xi) + g(X) \tag{8}$$

where $g : T(M) \rightarrow T^*(M)$ is identified to the bemolle musical isomorphism defined by:

$$g(X)(Y) = g(X,Y), \tag{9}$$

in block matrix form, is:

$$J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}. \tag{10}$$

Let $\nabla$ be a connection on $M$ and let $[\ ,\ ]_{\nabla}$ be the bracket on $C^{\infty}(E)$ defined by $\nabla$, the following holds:

**Lemma 4.** ([14]) *Let $J : E \rightarrow E$ be a generalized complex structure on $M$ and let*

$$N^{\nabla}(J) : C^{\infty}(E) \times C^{\infty}(E) \rightarrow C^{\infty}(E) \tag{11}$$

*defined by*:

$$N^{\nabla}(J)(\sigma,\tau) = [J\sigma, J\tau]_{\nabla} - J\,[J\sigma,\tau]_{\nabla} - J\,[\sigma,J\tau]_{\nabla} - [\sigma,\tau]_{\nabla} \tag{12}$$

*for all $\sigma,\tau \in C^{\infty}(E)$; $N^{\nabla}(J)$ is a skew symmetric tensor.*

**Definition 5.** $N^{\nabla}(J)$ is called the *Nijenhuis tensor of $J$ with respect to $\nabla$.*

**Definition 6.** Let $J : E \rightarrow E$ be a generalized complex structure on $M$, $J$ is said to be $\nabla-$*integrable* if $N^{\nabla}(J) = 0$.

**Proposition 7.** ([14]) *Let $\nabla$ be a torsion free connection on $M$ and let $J^g = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$ be the generalized complex structure on $M$ defined by a pseudo-Riemannian metric $g$, $J^g$ is $\nabla-$integrable if and only if $g$ is a Codazzi tensor, that is for all $X,Y \in C^{\infty}(T(M))$ we have:*

$$(\nabla_X g)Y = (\nabla_Y g)\,X. \tag{13}$$

A direct computation gives the following:

**Proposition 8.** *Let $\{x_1,...,x_n\}$ be local coordinates on $M$, let $\left\{\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n}\right\}$ be the corresponding local frame for $T(M)$, let $g_{ij} = g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)$ and $\{\Gamma_{ik}^l\}$ be the Cristoffel symbols of $\nabla$, then $g$ is a Codazzi tensor if and only if for all $i,j,k = 1,...,n$ :*

$$\frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_j} = \sum_{l=1}^{n}\left(\Gamma_{ik}^l g_{lj} - \Gamma_{jk}^l g_{li}\right). \tag{14}$$

Examples with non parallel $g$ can be found in the context of Hessian manifolds, their description will be the object of section 4.

## 2.2 Complex Lie algebroids and bialgebroids

The concept of Lie algebroid was introduced by Pradines in [16]; Lie bialgebroids were introduced by Mackenzie and Xu in [11] to encode the compatibility condition of a pair of two Lie algebroids in duality.

Here we first recall the definition of complex Lie algebroid:

**Definition 9.** A *complex Lie algebroid* is a complex vector bundle $L$ over a smooth real manifold $M$ such that: a Lie bracket $[\ ,\ ]$ is defined on $C^\infty(L)$, a smooth bundle map $\rho : L \to T(M)$, called *anchor*, is defined and, for all $\sigma, \tau \in C^\infty(L)$, for all $f \in C^\infty(M)$ the following conditions hold:

1. $\rho([\sigma,\tau]) = [\rho(\sigma), \rho(\tau)]$
2. $[f\sigma,\tau] = f([\sigma,\tau]) - (\rho(\tau)(f))\,\sigma.$

We now recall the definition of complex Lie bialgebroid.

Let $L$ and its dual vector bundle $L^*$ be Lie algebroids; on sections of $\wedge L$, respectively $\wedge L^*$, the *Schouten bracket* is defined by:

$$[\ ,\ ]_L : C^\infty(\wedge^p L) \times C^\infty(\wedge^q L) \longrightarrow C^\infty\left(\wedge^{p+q-1} L\right)$$

$$[X_1 \wedge ... \wedge X_p, Y_1 \wedge ... \wedge Y_q]_L =$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_i, Y_j]_L \wedge X_1 \wedge ..\overset{\widehat{i}}{.}.. \wedge X_p \wedge Y_1 \wedge ..\overset{\widehat{j}}{.}.. \wedge Y_q$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L)$

$$[X,f]_L = -[f,X]_L = \rho(X)(f);$$

respectively, by:

$$[\ ,\ ]_{L^*} : C^\infty(\wedge^p L^*) \times C^\infty(\wedge^q L^*) \longrightarrow C^\infty\left(\wedge^{p+q-1} L^*\right)$$

$$\left[X_1^* \wedge ... \wedge X_p^*, Y_1^* \wedge ... \wedge Y_q^*\right]_{L^*} =$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} \left[X_i^*, Y_j^*\right]_{L^*} \wedge X_1^* \wedge ..\overset{\widehat{i}}{.}.. \wedge X_p^* \Lambda Y_1^* \wedge ..\overset{\widehat{j}}{.}.. \wedge Y_q^*$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L^*)$

$$[X,f]_{L^*} = -[f,X]_{L^*} = \rho(X)(f).$$

Moreover the *exterior derivatives* $d$ and $d_*$ associated with the Lie algebroid structure of $L$ and $L^*$ are defined respectively by:

$$d : C^\infty(\wedge^p L^*) \longrightarrow C^\infty\left(\wedge^{p+1} L^*\right)$$

$$(d\alpha)(\sigma_0, ..., \sigma_p) =$$
$$= \sum_{i=0}^{p} (-1)^i \rho(\sigma_i)\, \alpha\left(\sigma_0, ..\overset{\widehat{i}}{.}.., \sigma_p\right) + \sum_{i<j} (-1)^{i+j} \alpha\left([\sigma_i, \sigma_j]_L, \sigma_0, ..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}.., \sigma_p\right)$$

for $\alpha \in C^\infty(\wedge^p L^*)$, $\sigma_0, ..., \sigma_p \in C^\infty(L)$,

and:

$$d_* : C^\infty(\wedge^p L) \longrightarrow C^\infty\left(\wedge^{p+1} L\right)$$

$$(d_*\alpha)(\sigma_0, ..., \sigma_p) =$$
$$= \sum_{i=0}^{p} (-1)^i \rho(\sigma_i)\, \alpha\left(\sigma_0, ..\overset{\widehat{i}}{.}.., \sigma_p\right) + \sum_{i<j} (-1)^{i+j} \alpha\left([\sigma_i, \sigma_j]_{L^*}, \sigma_0, ..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}.., \sigma_p\right)$$

for $\alpha \in C^\infty(\wedge^p L)$, $\sigma_0, ..., \sigma_p \in C^\infty(L^*)$.

**Definition 10.** A *complex Lie bialgebroid* is a pair of complex dual Lie algebroids $(L, L^*)$ such that the differential $d_*$ is a derivation of $(C^\infty(\wedge L), [\ ,\ ]_L)$, that is the following compatibility condition is satisfied:

$$d_*[\sigma, \tau]_L = [d_*\sigma, \tau]_L + [\sigma, d_*\tau]_L \tag{15}$$

for $\sigma$, $\tau \in C^\infty(L)$.

The following facts are well known:

**Proposition 11.** ([9]) *In a Lie bialgebroid* $(L, L^*)$, $d_*$ *is a derivation of the graded Lie algebra* $(C^\infty(\wedge L), [\ ,\ ]_L)$, *and* $d$ *is a derivation of* $(C^\infty(\wedge L^*), [\ ,\ ]_{L^*})$.

**Proposition 12.** ([9]) *Let* $(L, L^*)$ *be a pair of Lie algebroids in duality; the following properties are equivalent*:

1. $(L, L^*)$ *is a Lie bialgebroid,*
2. $d_*$ *is a derivation of* $(C^\infty(\wedge L), [\ ,\ ]_L)$,
3. $d$ *is a derivation of* $(C^\infty(\wedge L^*), [\ ,\ ]_{L^*})$,
4. $(L^*, L)$ *is a Lie bialgebroid.*

In the following section we will define natural Lie algebroids and bialgebroids in the context of generalized geometry.

## 3 Generalized $\overline{\partial}$-operator associated to $\mathbf{J}^g$

Let $(M,g)$ be a pseudo-Riemannian manifold and let $J^g$ be the generalized complex structure on $M$ defined by $g$, let

$$E^{\mathbb{C}} = (T(M) \oplus T^*(M)) \otimes \mathbb{C}$$

be the complexified generalized tangent bundle. The splitting in $\pm i$ eigenspaces of $J^g$ is denoted by:

$$E^{\mathbb{C}} = E^{1,0}_{J^g} \oplus E^{0,1}_{J^g} \tag{16}$$

with

$$E^{0,1}_{J^g} = \overline{E^{1,0}_{J^g}}. \tag{17}$$

A direct computation gives:

$$E^{1,0}_{J^g} = \{Z - ig(Z) \mid Z \in T(M) \otimes \mathbb{C}\}, \tag{18}$$

and, for any linear connection $\nabla$, the following holds:

**Lemma 13.** *$E^{1,0}_{J^g}$ and $E^{0,1}_{J^g}$ are $[\ ,\ ]_\nabla$ −involutive if and only if $N^\nabla(J^g) = 0$.*

**Proof.** Let $Z, W \in T(M) \otimes \mathbb{C}$, then we have:

$$[Z \pm ig(Z), W \pm ig(W)]_\nabla \mp iJ^g [Z \pm ig(Z), W \pm ig(W)]_\nabla =$$

$$= -(I \mp J^g) N^\nabla (J^g)(Z,W).$$

□

Moreover:

**Lemma 14.** *If $J^g$ is $\nabla$-integrable then Jacobi identity holds for $[\ ,\ ]_\nabla$ on $E^{1,0}_{J^g}$ and $E^{0,1}_{J^g}$.*

**Proof.** Let $Z, W, V \in T(M) \otimes \mathbb{C}$, then we have:

$$Jac\left[[Z \pm ig(Z), W \pm ig(W)]_\nabla, V \pm ig(V)\right]_\nabla =$$

$$= Jac\left[[Z,W], V\right] \pm ig\left(Jac\left[[Z,W], V\right]\right) = 0$$

where $Jac$ denotes *Jacobiator*, that is:

$$Jac\left[[\alpha,\beta]_\nabla, \gamma\right]_\nabla = \left[[\alpha,\beta]_\nabla, \gamma\right]_\nabla + \left[[\beta,\gamma]_\nabla, \alpha\right]_\nabla + \left[[\gamma,\alpha]_\nabla, \beta\right]_\nabla . \ \square$$

In particular we get:

**Proposition 15.** *If $J^g$ is $\nabla$-integrable then $E^{1,0}_{J^g}$ and $E^{0,1}_{J^g}$ are complex Lie algebroids.*

The following holds:

**Proposition 16.** *The natural symplectic structure on $E$ defines a canonical isomorphism between $E^{0,1}_{J^g}$ and the dual bundle of $E^{1,0}_{J^g}$, $\left(E^{1,0}_{J^g}\right)^*$.*

**Proof.** Let $Z, W \in T(M) \otimes \mathbb{C}$, we define:

$$(W + ig(W))(Z - ig(Z)) = (W + ig(W), Z - ig(Z)).$$

We get:

$$(W + ig(W))(Z - ig(Z)) = -ig(W, Z).$$

$\square$

The canonical isomorphism between $E^{0,1}_{J^g}$ and the dual bundle $\left(E^{1,0}_{J^g}\right)^*$ allows us to define the $\overline{\partial}_{J^g} - operator$ associated to the complex structure $J^g$ as in the following:

let $f \in C^\infty(M)$ and let $df \in C^\infty(T^*(M)) \hookrightarrow C^\infty(T(M) \oplus T^*(M))$, we pose

$$\overline{\partial}_{J^g} f = 2(df)^{0,1} = df + iJ^g df$$

or:

$$\overline{\partial}_{J^g} f = df - ig^{-1}(df);$$

moreover we define:

$$\overline{\partial}_{J^g} : C^\infty\left(E^{0,1}_{J^g}\right) \to C^\infty\left(\wedge^2\left(E^{0,1}_{J^g}\right)\right)$$

via the natural isomorphism

$$\left(E^{1,0}_{J^g}\right)^* \simeq E^{0,1}_{J^g}$$

as:

$$\overline{\partial}_{J^g} : C^\infty\left(\left(E^{1,0}_{J^g}\right)^*\right) \to C^\infty\left(\wedge^2\left(E^{1,0}_{J^g}\right)^*\right)$$

$$\left(\overline{\partial}_{J^g}\alpha\right)(\sigma, \tau) = \rho(\sigma)\alpha(\tau) - \rho(\tau)\alpha(\sigma) - \alpha([\sigma, \tau]_\nabla)$$

for $\alpha \in C^\infty\left(\left(E^{1,0}_{J^g}\right)^*\right)$, $\sigma, \tau \in C^\infty\left(E^{1,0}_{J^g}\right)$.

In general:

$$\overline{\partial}_{J^g} : C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)^*\right) \to C^\infty\left(\wedge^{p+1}\left(E^{1,0}_{J^g}\right)^*\right)$$

is defined by:

$$\left(\overline{\partial}_{J^g}\alpha\right)(\sigma_0,...,\sigma_p) =$$

$$= \sum_{i=0}^{p}(-1)^i\rho(\sigma_i)\,\alpha\left(\sigma_0,..\overset{\widehat{i}}{.}..,\sigma_p\right) + \sum_{i<j}(-1)^{i+j}\alpha\left([\sigma_i,\sigma_j]_\nabla,\sigma_0,..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}..,\sigma_p\right)$$

for $\alpha \in C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)^*\right)$, $\sigma_0,...,\sigma_p \in C^\infty\left(E^{1,0}_{J^g}\right)$.

**Definition 17.** $\overline{\partial}_{J^g}$ is called *generalized* $\overline{\partial}-$*operator* of $(M,g,\nabla)$ or *generalized* $\overline{\partial}_{J^g}-$ *operator.*

We have immediately that $\overline{\partial}_{J^g}$ is the exterior derivative, $d_L$, of the Lie algebroid $L = E^{1,0}_{J^g}$. Moreover the exterior derivative $d_{L^*}$ of $L^* = \left(E^{1,0}_{J^g}\right)^*$ is given by the operator $\partial_{J^g}$ defined by:

$$\partial_{J^g} : C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)\right) \to C^\infty\left(\wedge^{p+1}\left(E^{1,0}_{J^g}\right)\right)$$

$$\left(\partial_{J^g}\sigma\right)(\alpha_0^*,...,\alpha_p^*) =$$

$$= \sum_{i=0}^{p}(-1)^i\rho(\alpha_i^*)\,\sigma\left(\alpha_0^*,..\overset{\widehat{i}}{.}..,\alpha_p^*\right) + \sum_{i<j}(-1)^{i+j}\sigma\left([\alpha_i^*,\alpha_j^*]_\nabla,\alpha_0^*,..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}..,\alpha_p^*\right)$$

for $\sigma \in C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)\right)$, $\alpha_0^*,...,\alpha_p^* \in C^\infty\left(\left(E^{1,0}_{J^g}\right)^*\right)$.

We get the following:

**Proposition 18.** *If* $J^g$ *is* $\nabla -$ *integrable then* $\left(\overline{\partial}_{J^g}\right)^2 = 0$ *and* $\left(\partial_{J^g}\right)^2 = 0$.

**Proof.** It follows from the fact that Jacobi identity holds on $E^{1,0}_{J^g}$ and $\left(E^{1,0}_{J^g}\right)^*$. $\square$

**Definition 19.** $\alpha \in C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)^*\right)$ is called *generalized holomorphic* if $\overline{\partial}_{J^g}\alpha = 0$.

We remark that $\overline{\partial}_{J^g}f = 0 \Longleftrightarrow df = 0$, so the generalized holomorphic condition for functions gives only constant functions on connected components of $M$.

**Proposition 20.** *Let* $\sigma = X + ig(X) \in E^{0,1}_{J^g}$ *then* $\overline{\partial}_{J^g}\sigma = 0$ *if and only if* $g(X)$ *is a d-closed 1-form.*

**Proof.** Let $X, Y \in T(M)\otimes\mathbb{C}$, we have:

$\left(\overline{\partial}_{J^g}\sigma\right)(Y - ig(Y), Z - ig(Z))$
$= Y(X + ig(X), Z - ig(Z)) - Z(X + ig(X), Y - ig(Y)) +$
$-(X + ig(X), [Y, Z] - ig([Y, Z]))$
$= i\{Yg(X, Z) + Zg(X, Y) + g(X, [Y, Z])\}$
$= -i(dg(X))(Y, Z).$ $\square$

In particular, by using a classical result in symplectic geometry, [12], we get:

**Proposition 21.** *Let $X \in T(M)\otimes\mathbb{C}$, then $\sigma = X + ig(X) \in E^{0,1}_{J^g}$ is $\overline{\partial}_{J^g}-$cloded if and only if $g(X)$ is a Lagrangian submanifold of $T^*(M)$ with the standard symplectic structure.*

# 4 Examples

As we remarked in section 2. Hessian manifolds appear naturally in this context and provide interesting examples. Their introduction was inspired by the Bergmann metric on bounded domains in $\mathbb{C}^n$ and are a very interesting topic, related to many other fields in mathematics and theoretical physics as, for example: Kähler and symplectic geometry, affine differential geometry, special manifolds, string theory and mirror symmetry, [2], [4], [5], [10], [17], [18].

We now recall the general definition of Hessian metric:

**Definition 22.** Let $(M, g, \nabla)$ be a pseudo-Riemannian manifold with a torsion free linear connection, $g$ is called of *Hessian type* if there exists $u \in C^\infty(M)$ such that $g = Hess(u) = \nabla^2 u$. $(M, g, \nabla)$ is called *Hessian (pseudo-Riemannian) manifold* if $g$ is of Hessian type.

We prove the following:

**Proposition 23.** *Let $(M, g, \nabla)$ be a Hessian (pseudo-Riemannian) manifold then $g$ is a Codazzi tensor if and only if $\nabla$ is flat.*

**Proof.** Let $\{x_1, ..., x_n\}$ be local coordinates on $M$, let $g = \nabla^2 u$, then:

$$g_{jk} = \frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{l=1}^n \Gamma^l_{jk} \frac{\partial u}{\partial x_l}$$

in particular $g$ is a Codazzi tensor if and only if for all $i, j, k = 1, ..., n$ :

$$\begin{aligned}
&\frac{\partial^3 u}{\partial x_j \partial x_k \partial x_i} - \sum_{l=1}^n \frac{\partial \Gamma^l_{jk}}{\partial x_i}\frac{\partial u}{\partial x_l} - \sum_{l=1}^n \Gamma^l_{jk}\frac{\partial^2 u}{\partial x_l \partial x_i} + \\
&-\frac{\partial^3 u}{\partial x_i \partial x_k \partial x_j} + \sum_{l=1}^n \frac{\partial \Gamma^l_{ik}}{\partial x_j}\frac{\partial u}{\partial x_l} + \sum_{l=1}^n \Gamma^l_{ik}\frac{\partial^2 u}{\partial x_l \partial x_j} \\
= \quad &\sum_{l=1}^n \left(\Gamma^l_{ik}\left(\frac{\partial^2 u}{\partial x_l \partial x_j} - \sum_{r=1}^n \Gamma^r_{lj}\frac{\partial u}{\partial x_r}\right) - \Gamma^l_{jk}\left(\frac{\partial^2 u}{\partial x_l \partial x_i} - \sum_{r=1}^n \Gamma^r_{li}\frac{\partial u}{\partial x_r}\right)\right),
\end{aligned}$$

or:
$$\sum_{l=1}^{n}\left(\frac{\partial\Gamma^l_{ik}}{\partial x_j}-\frac{\partial\Gamma^l_{jk}}{\partial x_i}+\sum_{r=1}^{n}\left(\Gamma^r_{ik}\Gamma^l_{rj}-\Gamma^r_{jk}\Gamma^l_{ri}\right)\right)\frac{\partial u}{\partial x_l}=0$$
and thus the statement. □

We remark that under the hypothesis of Proposition 23 $g$ is Codazzi if and only if $(M,g,\nabla)$ is an affine Hessian manifold. Hessian manifolds are more general than affine manifolds, [10], however in the following we will consider affine Hessian manifolds.

Moreover the following is well known:

**Proposition 24.** ([3]) *Let $(\mathbb{R}^n,\nabla)$ be the euclidean n-dimensional space with $\nabla$ Levi Civita connection of the standard flat metric and let $g$ be a symmetric tensor of type $(2,0)$, then $g$ is a Codazzi tensor if and only if there exists $u\in C^\infty(\mathbb{R}^n)$ such that $g=Hess(u)$.*

We have the following:

**Proposition 25.** *Let $(M,g,\nabla)$ be an affine Hessian (pseudo-Riemannian) manifold, let $\{x_1,...,x_n\}$ be affine local coordinates, let $\left\{\frac{\partial}{\partial x_1},....,\frac{\partial}{\partial x_n}\right\}$ and $\{dx_1,...,dx_n\}$ be corresponding local frames for $T(M)$ and $T^*(M)$ respectively, let $g_{ij}=g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)$ then for all $k=1,...,n$ the local section*
$$\sigma_k=\frac{\partial}{\partial x_k}+i\sum_{l=1}^{n}g_{kl}dx_l\in C^\infty\left(E^{0,1}_{J^g}\right)$$
*is $\overline{\partial}_{J^g}-$cloded.*

**Proof.** Let $u\in C^\infty(M)$ such that $g=\nabla^2u$, as $dx_1,...,dx_n$ are $\nabla-$parallel, we have $\Gamma^l_{jk}=0$, then:
$$g_{jk}=\frac{\partial^2u}{\partial x_j\partial x_k},$$
in particular:
$$g\left(\frac{\partial}{\partial x_k}\right)=\sum_{l=1}^{n}g_{kl}dx_l=\sum_{l=1}^{n}\frac{\partial^2u}{\partial x_k\partial x_l}dx_l$$
and:
$$d\left(\sum_{l=1}^{n}g_{kl}dx_{_l}\right)=\sum_{l,j=1}^{n}\frac{\partial g_{kl}}{\partial x_j}dx_j\wedge dx_l=\sum_{j<l}\left(\frac{\partial g_{kl}}{\partial x_j}-\frac{\partial g_{kj}}{\partial x_l}\right)dx_j\wedge dx_l=0$$
then the statement. □

Moreover we can prove the following:

**Proposition 26.** *Let $(M, g, \nabla)$ be an affine Hessian manifold then the pair of complex dual Lie algebroids $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ is a Lie bialgebroid if and only if $g$ is constant.*

**Proof.** Let $\{x_1, ..., x_n\}$ be affine local coordinates, let $\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right\}$ and $\{dx_1, ..., dx_n\}$ be corresponding local frames for $T(M)$ and $T^*(M)$ respectively, let $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ and, for all $k = 1, ..., n$, let $\sigma_k = \frac{\partial}{\partial x_k} - i\sum_{l=1}^{n} g_{kl}dx_l \in C^\infty\left(E^{1,0}_{J^g}\right)$, we have:

$$[\sigma_k, \sigma_r]_\nabla = -i\sum_{s=1}^{n} \nabla_{\frac{\partial}{\partial x_k}} g_{rs}dx_s + i\sum_{l=1}^{n} \nabla_{\frac{\partial}{\partial x_r}} g_{kl}dx_l$$

$$= -i\left(\sum_{s=1}^{n} \frac{\partial g_{rs}}{\partial x_k}dx_s + \sum_{s=1}^{n} g_{rs}\left(\nabla_{\frac{\partial}{\partial x_k}} dx_s\right) - \sum_{l=1}^{n} \frac{\partial g_{kl}}{\partial x_r}dx_l - \sum_{l=1}^{n} g_{kl}\left(\nabla_{\frac{\partial}{\partial x_r}} dx_l\right)\right)$$

$$= -i\sum_{s=1}^{n}\left(\frac{\partial g_{rs}}{\partial x_k} - \frac{\partial g_{ks}}{\partial x_r}\right)dx_s = 0.$$

On the other hand:

$\sigma_k = \frac{\partial}{\partial x_k} - i\sum_{l=1}^{n} g_{kl}dx_l$ *is* $\partial_{J^g}-$closed, then we have immediately:

$$\partial_{J^g}[\sigma_k, \sigma_r]_\nabla = [\partial_{J^g}\sigma_k, \sigma_r]_\nabla + [\sigma_k, \partial_{J^g}\sigma_r]_\nabla = 0;$$

moreover, for all $f \in \wedge^0\left(E^{1,0}_{J^g}\right)$, we have:

$$\partial_{J^g}[f, \sigma_k]_\nabla - [\partial_{J^g} f, \sigma_k]_\nabla + [f, \partial_{J^g}\sigma_k]_\nabla$$

$$= \partial_{J^g}[f, \sigma_k]_\nabla - [\partial_{J^g} f, \sigma_k]_\nabla$$

$$= \partial_{J^g}(-\rho(\sigma_k) f) - \left[df + ig^{-1}df, \rho(\sigma_k) - ig\rho(\sigma_k)\right]_\nabla = 0$$

if and only if:

$$\begin{cases} d\left(\frac{\partial f}{\partial x_k}\right) & = & \nabla_{\frac{\partial}{\partial x_k}} df - \nabla_{g^{-1}df}\, g\left(\frac{\partial}{\partial x_k}\right) \\ g^{-1}d\left(\frac{\partial}{\partial x_k}\right) & = & \nabla_{\frac{\partial}{\partial x_k}} g^{-1}df - \nabla_{g^{-1}df}\frac{\partial}{\partial x_k} \end{cases}$$

or:

$$\begin{cases} d\left(\frac{\partial f}{\partial x_k}\right) & = & \nabla_{\frac{\partial}{\partial x_k}} df - \nabla_{g^{-1}df}\, g\left(\frac{\partial}{\partial x_k}\right) \\ g^{-1}\left(\nabla_{\frac{\partial}{\partial x_k}} df - \nabla_{g^{-1}df}\, g\left(\frac{\partial}{\partial x_k}\right)\right) & = & \nabla_{\frac{\partial}{\partial x_k}} g^{-1}df - \nabla_{g^{-1}df}\frac{\partial}{\partial x_k} \end{cases}. \quad (19)$$

The second condition in (19) is a consequence of $\nabla$-integrability of $J^g$, then the Lie bialgebroid condition (15) is reduced to:

$$d\left(\frac{\partial f}{\partial x_k}\right) = \nabla_{\frac{\partial}{\partial x_k}} df - \nabla_{g^{-1}df}\, g\left(\frac{\partial}{\partial x_k}\right) \tag{20}$$

or, by using Einstein's convention on repeated indices:

$$g^{hs}\frac{\partial g_{ki}}{\partial x_s} = 0$$

then, for all $i, k, s$:

$$\frac{\partial g_{ki}}{\partial x_s} = 0 \tag{21}$$

and thus the statement. $\square$

In particular we can reformulate Proposition 26 as the following:

**Proposition 27.** *Let $(M, g, \nabla)$ be an affine Hessian manifold then the pair of complex dual Lie algebroids $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ is a Lie bialgebroid if and only if $\nabla$ is the Levi Civita connection of $g$.*

We remark that the generalized $\overline{\partial}_{J^g} -$ *operator* introduced in this paper,

$$\overline{\partial}_{J^g} : C^\infty\left(E^{0,1}_{J^g}\right) \to C^\infty\left(\wedge^2\left(E^{0,1}_{J^g}\right)\right),$$

and the $\overline{\partial}_J -$ *operator* for Hitchin's generalized complex structures,

$$\overline{\partial}_J : C^\infty\left(E^{0,1}_J\right) \to C^\infty\left(\wedge^2\left(E^{0,1}_J\right)\right),$$

are defined formally in the same way, (see ([7]), Section 3.3). Here we use $[\ ,\ ]_\nabla$, restricted to sections of $E^{0,1}_{J^g}$, instead of the Courant bracket, restricted to sections of $E^{0,1}_J$, and the standard symplectic form instead of the standard pseudo-Euclidean metric on $T(M) \oplus T^*(M)$, in the identifications $E^{0,1}_{J^g} \cong \left(E^{0,1}_{J^g}\right)^*$ and $E^{0,1}_J \cong \left(E^{0,1}_J\right)^*$ respectively. However Proposition 27 shows different behavior of the two operators regarding Lie bialgebroid structure of $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ and $\left(E^{1,0}_J, \left(E^{1,0}_J\right)^*\right)$, since a generalized complex structure in Hitchin's sense always induces a Lie bialgebroid structure.

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